# Optimality of Received Energy in Decision Fusion Over Rayleigh Fading Diversity MAC With Non-Identical Sensors

Domenico Ciuonzo, Gianmarco Romano, and Pierluigi Salvo Rossi

***Abstract*—Received-energy test for non-coherent decision fusion over a Rayleigh fading multiple access channel (MAC) without diversity was recently shown to be optimum in the case of conditionally mutually independent and identically distributed (i.i.d.) sensor decisions under specific conditions [C. R. Berger, M. Guerriero, S. Zhou, and P. Willett, "PAC vs. MAC for Decentralized Detection Using Noncoherent Modulation," *IEEE Trans. Signal Process.*, vol. 57, no. 9, pp. 3562–2575, Sep. 2009], [F. Li, J. S. Evans, and S. Dey, "Decision Fusion Over Noncoherent Fading Multiaccess Channels," *IEEE Trans. Signal Process.*, vol. 59, no. 9, pp. 4367–4380, Sep. 2011]. Here, we provide a twofold generalization, allowing sensors to be non identical on one hand and introducing diversity on the other hand. Along with the derivation, we provide also a general tool to verify optimality of the received energy test in scenarios with correlated sensor decisions. Finally, we derive an analytical expression of the effect of the diversity on the large-system performances, under both individual and total power constraints.**

***Index Terms*—Distributed detection, energy detection, MIMO, wireless sensor networks.**

## I. Introduction

Starting from classical distributed detection [3], large efforts in the recent literature have been devoted to the implementation of distributed detection in wireless sensor networks (WSNs) [4]–[6]. Local decisions in a WSN are usually transmitted to a decision fusion center (DFC) in order to improve reliability of geographically distributed sensing through central processing. Common system architectures make reference to the availability of parallel (non-interfering) channels from the sensors to the DFC [7]–[9]. However, more sophisticated setups have been investigated, where the intrinsically interfering nature of the wireless channel is exploited and not combated [1], [10], [11].

Recently, in [1] and [2], the received-energy test was studied for non-coherent decision fusion over a multiple access channel (MAC). More specifically, in [1] the received energy was claimed as optimal for the no-diversity case with conditionally (given the phenomenon) mutually independent and identically distributed (i.i.d.) sensor local decisions, as long as the probability of false alarm of the generic sensor is lower than the corresponding probability of detection. Also, analytical performances of the received-energy test in the diversity scenario were derived. However, optimality property of the test was not investigated. The optimality of the test for the no-diversity case with conditionally i.i.d. sensor local decisions was proven in [2]. Only the case with sensors whose probability of false alarm is lower than the corresponding probability of detection was considered. Nonetheless, the diversity case was still ignored in the optimality analysis.

Manuscript received April 12, 2012; revised June 16, 2012 and September 17, 2012; accepted October 02, 2012. Date of publication October 09, 2012; date of current version December 04, 2012. The associate editor coordinating the review of this manuscript and approving it for publication was Prof. Shuguang (Robert) Cui.

The authors are with the Department of Information Engineering, Second University of Naples, 81031 Aversa (CE), Italy (e-mail: domenico.ciuonzo@unina2.it; gianmarco.romano@unina2.it; pierluigi.salvorossi@ unina2.it).

Color versions of one or more of the figures in this paper are available online at http://ieeexplore.ieee.org.

Digital Object Identifier 10.1109/TSP.2012.2223694

The main contributions of this correspondence are:

- a rigorous proof of the *optimality* of the received-energy test for non-coherent decision fusion[1] over a Rayleigh fading MAC with *arbitrary order of diversity* and with conditionally mutually independent *but non identically distributed* (i.n.i.d.) sensor decisions, as long as *each* sensor probability of false alarm is lower than the correspondent probability of detection;
- as a side result, a sufficient condition on the log-likelihood ratio (LLR) of the number of active sensors suited for testing received energy optimality in scenarios with correlated local decisions;
- analytical derivation of large-system performances for conditionally i.i.d. sensor local decisions as a function of the order of diversity, where two different scenarios are considered: (a) sensors with an individual power constraint (IPC); (b) sensors with a total power constraint (TPC).

It is worth noting that in [11] a different scenario was analyzed, where: (*i*) conditionally i.i.d. sensor decisions were considered, and (*ii*) instantaneous channel state information (CSI) at the DFC was assumed. The focus was on the performance analysis of several sub-optimal fusion rules in terms of complexity, required knowledge, probability of detection and false alarm.

The paper is organized as follows: Section II introduces the system model; in Section III we present the main results of this correspondence, while in Section IV we draw some concluding remarks; proofs are confined to Appendices.

*Notation*—Lower-case (resp. Upper-case) bold letters denote vectors (resp. matrices), with $a_n$ (resp. $a_{n,m}$) representing the $n$th (resp. the $(n,m)$th) element of the vector $\boldsymbol{a}$ (resp. matrix $\boldsymbol{A}$); upper-case calligraphic letters, e.g., $\mathcal{A}$, denote discrete and finite sets; $\boldsymbol{I}_N$ denotes the $N \times N$ identity matrix; $\mathbf{0}_N$ (resp. $\mathbf{1}_N$) denotes the null (resp. ones) vector of length $N$; $\mathbb{E}\{\cdot\}$, $(\cdot)^t$, $(\cdot)^\dagger$, $\Re(\cdot)$, $\Im(\cdot)$ and $\|\cdot\|$ denote expectation, transpose, conjugate transpose, real part, imaginary part and Frobenius norm operators; $P(\cdot)$ and $p(\cdot)$ are used to denote probability mass functions (pmf) and probability density functions (pdf), while $P(\cdot\,|\,\cdot)$ and $p(\cdot\,|\,\cdot)$ their corresponding conditional counterparts; $\mathcal{N}_{\mathbb{C}}(\boldsymbol{\mu},\boldsymbol{\Sigma})$ (resp. $\mathcal{N}(\boldsymbol{\mu},\boldsymbol{\Sigma})$) denotes a circular symmetric complex (resp. real) normal distribution with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$, $\mathcal{B}(k,p)$ denotes a binomial distribution of $k$ trials with probability of success $p$ and $\chi^2_L$ denotes a chi-square distribution with $L$ degrees of freedom; $(a * b)(\ell)$ denotes the convolution between series $a(\ell)$ and $b(\ell)$; finally the symbols $\sim$ and $\xrightarrow{d}$ mean "distributed as" and "convergence in distribution".

## II. System Model

### A. WSN Modeling

We consider a distributed binary hypothesis test, where $K$ sensors are used to discriminate between the hypotheses of the set $\mathcal{H} = \{H_0, H_1\}$, representing, not necessarily, the absence $(H_0)$ or the presence $(H_1)$ of a specific target of interest. The $k$th sensor, $k \in \mathcal{K} \triangleq \{1,2,\ldots,K\}$, takes a binary local decision $d_k \in \mathcal{H}$ about the observed phenomenon on the basis of its own measurements.

Each decision $d_k$ is mapped to a symbol $x_k \in \mathcal{X} = \{0,1\}$ representing an On-Off Keying (OOK) modulation: without loss of generality we assume that $d_k = H_i$ maps into $x_k = i$, $i \in \{0,1\}$. The quality of the $k$th sensor decisions is characterized by the conditional probabilities $P(x_k\,|\,H_j)$. More specifically, we denote $P_{D,k} \triangleq$

[1]Although, energy receiver and non-coherent are not synonyms, in the paper we will confuse them. In the related literature, such a misuse is common due to the fact that the energy detector is the default receiver adopted for non-coherent decision fusion.

$P(x_k = 1 \mid H_1)$ and $P_{F,k} \triangleq P(x_k = 1 \mid H_0)$, respectively the probability of detection and false alarm of the $k$th sensor.

The sensors communicate with the DFC over a wireless flat-fading MAC, modeled through i.i.d. Rayleigh fading coefficients with equal mean power. The DFC employs an $N$-diversity approach in order to combat signal attenuation due to small-scale fading of the wireless medium. The diversity can be accomplished with time, frequency, code or antenna diversity (as recently proposed in [10], [12]). Statistical CSI is assumed at the DFC, i.e., only the pdf of each fading coefficient is available.

We denote: $y_n$ the received signal at the $n$th diversity branch of the DFC after matched filtering and sampling; $h_{n,k} \sim \mathcal{N}_{\mathbb{C}}(0, \sigma_h^2)$ the fading coefficient between the $k$th sensor and the $n$th diversity branch of the DFC[2]; $w_n$ the additive white Gaussian noise at the $n$th diversity branch of the DFC. The vector model at the DFC is the following:

$$\boldsymbol{y} = \boldsymbol{H}\boldsymbol{x} + \boldsymbol{w} \tag{1}$$

where $\boldsymbol{y} \in \mathbb{C}^N$, $\boldsymbol{H} \in \mathbb{C}^{N \times K}$, $\boldsymbol{x} \in \mathcal{X}^K$, $\boldsymbol{w} \sim \mathcal{N}_{\mathbb{C}}(\boldsymbol{0}_N, \sigma_w^2 \boldsymbol{I}_N)$ are the received signal vector, the channel matrix, the transmitted signal vector and the noise vector, respectively. Finally, we define the random variable $\ell \triangleq \ell(\boldsymbol{x}) = \sum_{k=1}^K x_k$, representing the number of active sensors and the set $\mathcal{L} \triangleq \{0, \ldots, K\}$ of possible realizations of $\ell$.

### B. LLR

The optimal test [3], [13] for the considered problem can be formulated as

$$\Lambda_{\text{opt}} \triangleq \ln \left[ \frac{p(\boldsymbol{y} \mid H_1)}{p(\boldsymbol{y} \mid H_0)} \right] \underset{\hat{H}=H_0}{\overset{\hat{H}=H_1}{\gtrless}} \gamma \tag{2}$$

where $\hat{H}$, $\Lambda_{\text{opt}}$ and $\gamma$ denote the estimated hypothesis, the Log-Likelihood-Ratio (LLR, i.e., the optimal fusion rule) and the threshold to which the LLR is compared to. The threshold $\gamma$ can be determined to assure a fixed system false-alarm rate (Neyman-Pearson approach) or can be chosen to minimize the probability of error (Bayes approach) [3], [13]. An explicit expression of the LLR from (2) is given by

$$\Lambda_{\text{opt}} = \ln \left[ \frac{\sum_{\ell=0}^K p(\boldsymbol{y} \mid \ell) P(\ell \mid H_1)}{\sum_{\ell=0}^K p(\boldsymbol{y} \mid \ell) P(\ell \mid H_0)} \right] \tag{3}$$

$$= \ln \left[ \frac{\sum_{\ell=0}^K \frac{1}{(\sigma_w^2 + \ell\sigma_h^2)^N} \exp\left(-\frac{\|\boldsymbol{y}\|^2}{\sigma_w^2 + \ell\sigma_h^2}\right) P(\ell \mid H_1)}{\sum_{\ell=0}^K \frac{1}{(\sigma_w^2 + \ell\sigma_h^2)^N} \exp\left(-\frac{\|\boldsymbol{y}\|^2}{\sigma_w^2 + \ell\sigma_h^2}\right) P(\ell \mid H_0)} \right] \tag{4}$$

where we have exploited the conditional independence of $\boldsymbol{y}$ from $H_i$ (given $\ell$).

In the case of conditionally (given $H_i$) i.i.d. sensor decisions ($(P_{D,k}, P_{F,k}) = (P_D, P_F), k \in \mathcal{K}$) we have that $\ell \mid H_1 \sim \mathcal{B}(K, P_D)$ and $\ell \mid H_0 \sim \mathcal{B}(K, P_F)$. Differently, when local sensor decisions are conditionally i.n.i.d. the pmfs $P(\ell \mid H_i)$ are represented by the more general *Poisson-Binomial* distribution [14]–[16], with expressions given by:

$$P(\ell \mid H_1) = \sum_{\boldsymbol{x}:\ell(\boldsymbol{x})=\ell} \prod_{k=1}^K (P_{D,k})^{x_k} \prod_{s=1}^K (1 - P_{D,s})^{(1-x_s)} \tag{5}$$

$$P(\ell \mid H_0) = \sum_{\boldsymbol{x}:\ell(\boldsymbol{x})=\ell} \prod_{k=1}^K (P_{F,k})^{x_k} \prod_{s=1}^K (1 - P_{F,s})^{(1-x_s)} \tag{6}$$

It is worth noting that (5) requires sums which are infeasible to compute in practice unless the number of sensors $K$ is small. For this reason different methods have been proposed in literature for its efficient evaluation. The alternatives include fast convolution of individual Bernoulli pmfs [14], recursive approaches [15] and a Discrete Fourier Transform (DFT) based computation [16].

## III. Optimality of Received Energy

As already stated in [1], the received energy $\psi \triangleq \|\boldsymbol{y}\|^2$ is a sufficient statistic for the LLR, since (3) depends on $\boldsymbol{y}$ only through $\psi$. However, *sufficiency* alone does not guarantee that the test

$$\psi \underset{\hat{H}=H_0}{\overset{\hat{H}=H_1}{\gtrless}} \gamma' \tag{7}$$

is equivalent to (2). As shown in [2], the test in (7) is optimal *iff* $\Lambda_{\text{opt}}(\psi)$ is a strictly increasing function of $\psi$. If this property is satisfied, the test in (2) is equivalent to (7) by simply setting $\gamma' = \Lambda_{\text{opt}}^{-1}(\gamma)$. For this purpose in the following we first introduce a general optimality test (in the form of a sufficient condition) which relates the pmfs $P(\ell \mid H_i)$, $H_i \in \mathcal{H}$, to assure that $\Lambda_{\text{opt}}(\psi)$ is strictly increasing in the case of an $N$-diversity MAC.

*Proposition 1:* A sufficient condition for $\Lambda_{\text{opt}}(\psi)$ to be a strictly increasing function of $\psi$ is given by:

$$\lambda(\ell) > \lambda(\ell - 1), \quad \ell \in \mathcal{L} \backslash \{0\} \tag{8}$$

where $\lambda(\ell) \triangleq \ln[\frac{P(\ell \mid H_1)}{P(\ell \mid H_0)}]$.

*Proof:* The proof is reported in Appendix A. ■

The above proposition states that strictly increasing property of $\lambda(\ell)$ assures optimality of the test in (7). We will refer to $\lambda(\ell)$ as the $\ell$-LLR hereinafter[3]. It is worth noting that such a sufficient condition is independent of the order of diversity $N$. Also, (8) depends on the WSN model only through the number of active sensors $\ell$ (given $H_i$) and does not require any specific assumption on $P(\boldsymbol{x} \mid H_i)$, e.g., conditional mutual independence of local sensor decisions, i.e., $P(\boldsymbol{x} \mid H_i) = \prod_{k=1}^K P(x_k \mid H_i)$. This means that (8) plays the role of a general property for received energy optimality, to be verified even in the case of conditionally correlated local sensor decisions.

In the simplest case of conditionally i.i.d. local sensor decisions, $(P_{D,k}, P_{F,k}) = (P_D, P_F)$, $k \in \mathcal{K}$, as assumed in [1], [2], the strictly increasing property of $\lambda(\ell)$, $\ell \in \mathcal{L}$, is equivalent to

$$\frac{\binom{K}{\ell} P_D^\ell (1 - P_D)^{K-\ell}}{\binom{K}{\ell} P_F^\ell (1 - P_F)^{K-\ell}} > \frac{\binom{K}{\ell - 1} P_D^{\ell-1} (1 - P_D)^{K-\ell+1}}{\binom{K}{\ell - 1} P_F^{\ell-1} (1 - P_F)^{K-\ell+1}}, \tag{9}$$

that reduces to $P_D > P_F$. This result, not only confirms theoretical findings for optimality of (7) when $N = 1$ as in [1], [2], but it also proves the optimality of the test over the *Diversity MAC* (i.e., $N > 1$) used in [1]; this result shows the effectiveness and simplicity of Proposition 1 w.r.t. the approach taken in [2].

Differently, when sensor decisions are conditionally i.n.i.d. (i.e., the case of a heterogeneous WSN), the following theorem generalizes the result in (9).

[2]It is worth noting that assuming an asymmetric model for channel coefficient statistics would be more realistic. However, this would make the results much more dependent on the specific scenario without adding any significant insight from a theoretical point of view. A symmetric model for channel coefficient statistics is assumed for a two-fold reason: on one side it can be considered as a starting point before analyzing more realistic application-dependent scenarios; on the other side a symmetric scenario could represent scenarios in which power control is considered.

[3]Note that we will not consider in (8) (and throughout the paper) the case $\ell = 0$ when testing $\ell$-LLR strictly increasing property, since $\lambda(-1)$ has no physical meaning.

*Theorem 1:* If $P(\boldsymbol{x}\,|\,H_i) = \prod_{k=1}^{K} P(x_k\,|\,H_i)$, $H_i \in \mathcal{H}$, and $P_{D,k} > P_{F,k}$, $k \in \mathcal{K}$, the $\ell$-LLR satisfies the strictly increasing property described in (8) and thus (7) is the optimal test when an $N$-diversity MAC is employed.

*Proof:* The proof is reported in Appendix B. ■

Theorem 1 states that under non identical sensors $(P_{D,k}, P_{F,k})$ and $N$-diversity MAC the received energy $\psi$ is again the optimal test. Also, Theorem 1 relies on a sufficient condition, i.e., specific WSN configurations not satisfying the assumption $P_{D,k} > P_{F,k}$, $k \in \mathcal{K}$, but still verifying (8) may exist. Although such a case is not "typically" of interest, since the condition $P_{D,k} \leq P_{F,k}$ for $k$th sensor is not realistic in practical scenarios (i.e., sensors operating under nominal conditions), it proves the robustness of the received energy in scenarios with some faulty (or byzantine) sensors[4].

We finally evaluate analytically the performances as the number of sensors goes large, in the case of conditionally i.i.d. sensors. Both IPC and TPC on the WSN and arbitrary diversity $N$ are considered here. This result generalizes [2], where no-diversity $(N = 1)$ and IPC assumptions were made in deriving formulas. We define

$$\boldsymbol{z} \triangleq \frac{1}{\sqrt{P_F K \sigma_h^2}} \left( \frac{\boldsymbol{H}\boldsymbol{x}}{\sqrt{N}} + \boldsymbol{w} \right) \tag{10}$$

where, compared to (1), $\frac{1}{\sqrt{P_F K \sigma_h^2}}$ is a merely scaling factor and $\boldsymbol{H}\boldsymbol{x}$ is replaced with $\frac{\boldsymbol{H}\boldsymbol{x}}{\sqrt{N}}$ in $\boldsymbol{z}$ in order to keep a fixed amount of average energy $\varepsilon \triangleq E\{\|\boldsymbol{H}\boldsymbol{x}\|^2\}$ w.r.t. $N$. Then we define the system probabilities of false alarm and detection, respectively $P_{F_0}$ and $P_{D_0}$, as:

$$P_{F_0} \triangleq P(\|\boldsymbol{z}\|^2 \geq \bar{\gamma}\,|\,H_0), \quad P_{D_0} \triangleq P(\|\boldsymbol{z}\|^2 \geq \bar{\gamma}\,|\,H_1), \tag{11}$$

Equations (10) and (11) hold for TPC scenario when replacing $\boldsymbol{z}$ with $\tilde{\boldsymbol{z}} \triangleq \frac{1}{\sqrt{P_F \sigma_h^2}} \left( \frac{\boldsymbol{H}\boldsymbol{x}}{\sqrt{KN}} + \boldsymbol{w} \right)$. In this case the average energy is kept fixed w.r.t. both $K$ and $N$.

*Theorem 2:* If $\sigma_h^2$ and $\sigma_w^2$ are finite, as $K \to +\infty$

$$\begin{aligned} \boldsymbol{z}\,|\,H_0 &\xrightarrow{d} \mathcal{N}_{\mathbb{C}}\left(\boldsymbol{0}_N, \frac{1}{N}\boldsymbol{I}_N\right), \\ \boldsymbol{z}\,|\,H_1 &\xrightarrow{d} \mathcal{N}_{\mathbb{C}}\left(\boldsymbol{0}_N, \frac{P_D}{P_F N}\boldsymbol{I}_N\right), \end{aligned} \tag{12}$$

$$\begin{aligned} \tilde{\boldsymbol{z}}\,|\,H_0 &\xrightarrow{d} \mathcal{N}_{\mathbb{C}}\left(\boldsymbol{0}_N, \frac{1}{\alpha_F}\frac{1}{N}\boldsymbol{I}_N\right), \\ \tilde{\boldsymbol{z}}\,|\,H_1 &\xrightarrow{d} \mathcal{N}_{\mathbb{C}}\left(\boldsymbol{0}_N, \frac{1}{\alpha_D}\frac{P_D}{P_F N}\boldsymbol{I}_N\right), \end{aligned} \tag{13}$$

where $\alpha_F \triangleq \frac{P_F \sigma_h^2}{P_F \sigma_h^2 + \sigma_w^2 N}$ and $\alpha_D \triangleq \frac{P_D \sigma_h^2}{P_D \sigma_h^2 + \sigma_w^2 N}$. Then the large-system $(P_{D_0-\mathrm{IPC}}^*, P_{F_0-\mathrm{IPC}}^*)$ are given by:

$$P_{F_0-\mathrm{IPC}}^*(\bar{\gamma}) = \exp\left(-\bar{\gamma}N\right) \times \sum_{n=0}^{N-1} \frac{1}{n!}\left(\bar{\gamma}N\right)^n; \tag{14}$$

$$P_{D_0-\mathrm{IPC}}^*(\bar{\gamma}) = \exp\left(-\frac{\bar{\gamma}N}{\frac{P_D}{P_F}}\right) \times \sum_{n=0}^{N-1} \frac{1}{n!}\left(\frac{\bar{\gamma}N}{\frac{P_D}{P_F}}\right)^n; \tag{15}$$

while $(P_{D_0-\mathrm{TPC}}^*, P_{F_0-\mathrm{TPC}}^*)$ are given by:

$$P_{F_0-\mathrm{TPC}}^*(\bar{\gamma}) = \exp\left(-\bar{\gamma}N\alpha_F\right) \times \sum_{n=0}^{N-1} \frac{1}{n!}\left(\bar{\gamma}N\alpha_F\right)^n; \tag{16}$$

$$P_{D_0-\mathrm{TPC}}^*(\bar{\gamma}) = \exp\left(-\frac{\bar{\gamma}N\alpha_D}{\frac{P_D}{P_F}}\right) \sum_{n=0}^{N-1} \frac{1}{n!}\left(\frac{\bar{\gamma}N\alpha_D}{\frac{P_D}{P_F}}\right)^n. \tag{17}$$

[4]A WSN with $K = 3$ sensors such that $(P_{D,1}, P_{F,1}) = (0.5, 0.05)$, $(P_{D,2}, P_{F,2}) = (0.4, 0.1)$ and $(P_{D,3}, P_{F,3}) = (0.3, 0.4)$ verifies the property in (8).

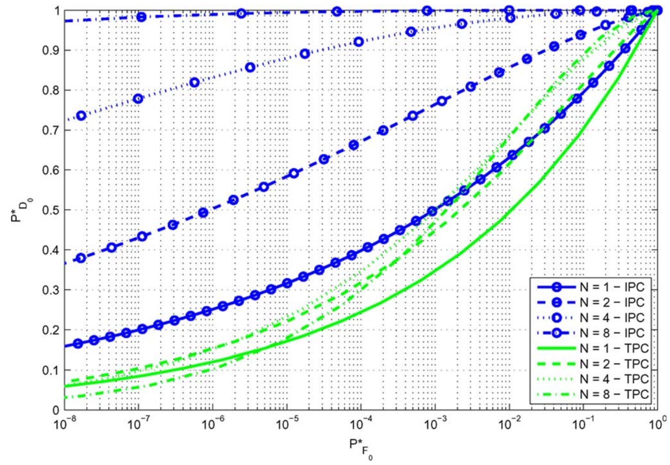


Fig. 1. Effect of diversity $N$ on large-system ROC $(P_{D_0}^*, P_{F_0}^*)$ under both IPC and TPC; WSN with sensor characteristics $(P_{D,k}, P_{F,k}) = (P_D, P_F) = (0.5, 0.05)$; $(\sigma_h^2/\sigma_w^2)_{\mathrm{dB}} = 15$.

*Proof:* The proof is reported in Appendix C for the IPC case; performance in the TPC scenario can be derived in a similar fashion. ■

As expected, if $N = 1$ the result of (14) and (15) coincides with the one given in ([2, Sec. IV]).

It is worth remarking that, in both IPC and TPC scenarios with diversity, Neyman-Pearson and Bayesian error exponents are zero (cfr. with [2]), because the large-system ROC can not be driven toward the point $(P_{D_0}^*, P_{F_0}^*) = (1, 0)$ by increasing the number of sensors, as long as the diversity $N$ is kept finite. This intuition is confirmed by the non-zero values assumed under an IPC and a TPC by the large-system J-Divergence, $J(p(\boldsymbol{z}\,|\,H_0), p(\boldsymbol{z}\,|\,H_1)) = N \times [(\frac{P_D}{P_F} + \frac{P_F}{P_D}) - 2]$, $J(p(\tilde{\boldsymbol{z}}\,|\,H_0), p(\tilde{\boldsymbol{z}}\,|\,H_1)) = N \times [(\frac{P_D \alpha_F}{P_F \alpha_D} + \frac{P_F \alpha_D}{P_D \alpha_F}) - 2]$, which represents a lower-bound for the system error probability [17], thus enforcing a zero Bayesian error exponent. Differently, the Neyman-Pearson error exponent is given by $\lim_{K\to+\infty} -\frac{\ln[1-P_{D_0}(\bar{\gamma},K)]}{K}$, under $P_{F_0}(\bar{\gamma},K) \leq \alpha$. If we choose $\bar{\gamma}_\alpha$ such that $P_{F_0}^*(\bar{\gamma}_\alpha) = \alpha$, then $\lim_{K\to+\infty} -\ln[1 - P_{D_0}(\bar{\gamma}_\alpha, K)] = -\ln[1 - P_{D_0}^*(\bar{\gamma}_\alpha)] < +\infty$, giving again a zero error exponent.

Note that the performance in TPC scenarios differ through the ratio $(\alpha_F/\alpha_D) < 1$ (cfr. (14) and (15) with (16) and (17)) which represents the *performance reduction factor* w.r.t. IPC scenarios. Note that $\alpha_F/\alpha_D$: (*i*) is an increasing function of the ratio $\frac{\sigma_h^2}{\sigma_w^2}$ (i.e., the received SNR), with limiting value equal to one; (*ii*) is a decreasing function of $N$, meaning a diverging separation in performance between IPC and TPC as $N$ increases.

The diversity affects in a significant way the large-system probabilities of detection and false alarm, under an IPC, by shifting the Receiver Operating Characteristic (ROC) toward the upper-left corner, as shown in Fig. 1, meaning a performance improvement. Differently, it can be seen how a different effect is present in the TPC case, where an increase of $N$ does not always coincide with performance improvement, but rather an optimal $N$, depending on $(P_D, P_F, \frac{\sigma_h^2}{\sigma_w^2})$, exists. This effect was already noticed in [1] and it is due to non-coherent combining loss of branch contributions.

Finally, in Figs. 2 and 3 we verify, through simulations, the convergence of the ROC to the large system expression $(K \to +\infty)$ given by (14) and (15) (resp. (16) and (17)), under IPC (resp. TPC). It is apparent that the convergence under the TPC is faster w.r.t. the IPC case, because in both cases the large system ROC expressions rely on the Gaussian approximation of the Gaussian mixture given by (3). For such a reason, for a given $K$, imposing a TPC on the WSN assures a better matching w.r.t. to the IPC case, since all the components of the mixture will be more concentrated.

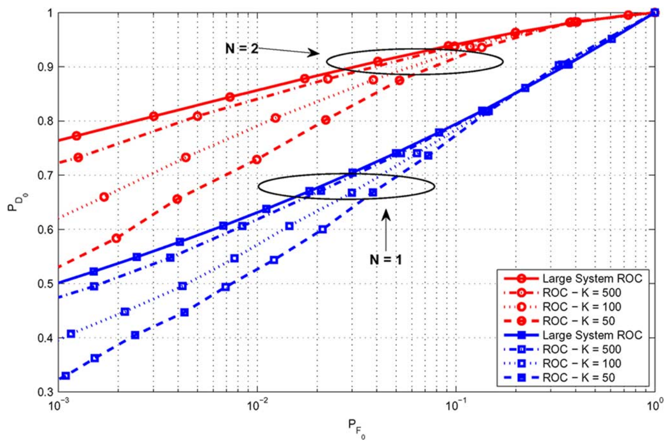


Fig. 2. ROC comparison: large system vs finite number of sensors ($K \in \{50, 100, 500\}$) under IPC. $N \in \{1, 2\}$, $(\sigma_h^2/\sigma_w^2)_{\mathrm{dB}} = 15$.

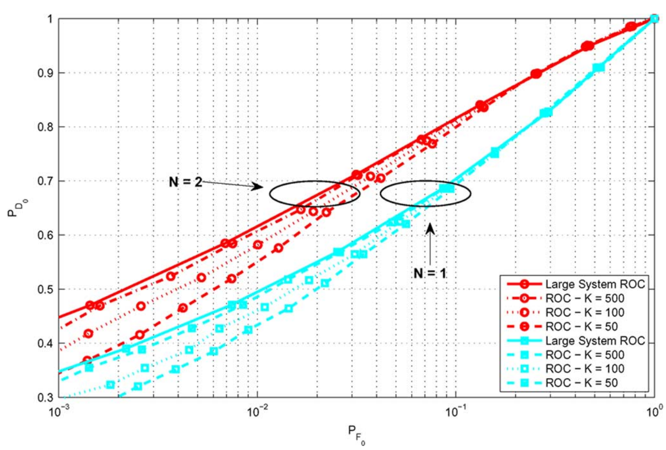


Fig. 3. ROC comparison: large system vs finite number of sensors ($K \in \{50, 100, 500\}$) under TPC. $N \in \{1, 2\}$, $(\sigma_h^2/\sigma_w^2)_{\mathrm{dB}} = 15$.

## IV. Concluding Remarks

In this correspondence we showed the optimality of the received-energy test for decision fusion performed over a non-coherent diversity MAC with conditionally i.n.i.d. sensor decisions. We derived a sufficient condition on the LLR of the number of active sensors which can be applied to test the received-energy optimality in WSN with conditionally correlated sensor decisions. Finally, we showed, through analytical results, how the diversity in a WSN with conditionally i.i.d. sensor decisions affects the large-system performance under both IPC and TPC.

## Appendix A
## Proof of Proposition 1

We prove in this section that (8) is a sufficient condition for the optimality of $\psi$ test. From (3), looking at the LLR as a function of $\psi$, we get:

$$\frac{\partial \Lambda_{\mathrm{opt}}(\psi)}{\partial \psi} = \frac{1}{\alpha(\psi)}\left[\sum_{\ell_1=0}^{K} \frac{\partial g(\psi,\ell_1)}{\partial \psi} P(\ell_1 \mid H_1) \sum_{\ell_2=0}^{K} g(\psi,\ell_2) P(\ell_2 \mid H_0)\right] - \frac{1}{\alpha(\psi)}\left[\sum_{\ell_2=0}^{K} \frac{\partial g(\psi,\ell_2)}{\partial \psi} P(\ell_2 \mid H_0 \sum_{\ell_1=0}^{K} g(\psi,\ell_1) P(\ell_1 \mid H_1)\right] \quad (18)$$

where we denoted $g(\psi,\ell) \triangleq \frac{1}{(\sigma_w^2+\ell\sigma_h^2)^N}\exp(-\frac{\psi}{\sigma_w^2+\ell\sigma_h^2})$ and $\alpha(\psi)$ indicates a positive function of $\psi$ (i.e., $\alpha(\psi) > 0$, $\forall \psi \in \mathbb{R}^+$). Strictly increasing property of LLR is guaranteed if $\frac{\partial \Lambda_{\mathrm{opt}}(\psi)}{\partial \psi} > 0$, $\forall \psi \in \mathbb{R}^+$, thus manipulations from (18) lead to

$$\sum_{\ell_1=1}^{K}\sum_{\ell_2=0}^{\ell_1-1} k(\ell_1,\ell_2)\left[\frac{\partial g(\psi,\ell_1)}{\partial \psi} g(\psi,\ell_2) - \frac{\partial g(\psi,\ell_2)}{\partial \psi} g(\psi,\ell_1)\right] > 0 \quad (19)$$

where $k(\ell_1,\ell_2) \triangleq [P(\ell_1 \mid H_1)P(\ell_2 \mid H_0) - P(\ell_2 \mid H_1)P(\ell_1 \mid H_0)]$. In deriving (19) we could express the double sums in (18) as a function only of the indices $\ell_1 > \ell_2$, since the term in bracket in (19) equals to zero when $\ell_1 = \ell_2$. Noting that $\frac{\partial g(\psi,\ell)}{\partial \psi} = -\frac{1}{(\sigma_w^2+\ell\sigma_h^2)} g(\psi,\ell)$ the condition is rewritten as

$$\sum_{\ell_1=1}^{K}\sum_{\ell_2=0}^{\ell_1-1} k(\ell_1,\ell_2) g(\psi,\ell_1) g(\psi,\ell_2)\left[\frac{\sigma_h^2(\ell_1-\ell_2)}{(\sigma_w^2+\ell_1\sigma_h^2)(\sigma_w^2+\ell_2\sigma_h^2)}\right] > 0 \quad (20)$$

Since both $g(\psi,\ell)$ and the term in square brackets are positive (note that indices in the sums are such that $\ell_1 > \ell_2$), the term $k(\ell_1,\ell_2)$ is responsible for the sign of each term in the sum. Then a *sufficient condition* for (20) is obtained assuming that each of those terms is positive. This is achieved if the following property holds

$$k(\ell_1,\ell_2) > 0, \quad \ell_1 > \ell_2. \quad (21)$$

It is easy to demonstrate that the condition $k(\ell,\ell-1) > 0$, $\ell \in \mathcal{L}\backslash\{0\}$, representing the strictly increasing property of $\ell$-LLR, i.e., $\frac{P(\ell \mid H_1)}{P(\ell \mid H_0)} > \frac{P(\ell-1 \mid H_1)}{P(\ell-1 \mid H_0)}$, is equivalent to (21). In fact (21) implies that $\ell$-LLR is strictly increasing; this is verified just substituting $\ell_2 = \ell_1 - 1$. Differently, we can show that $\ell$-LLR strictly increasing property implies (21) by constructing the chain of inequalities $\frac{P(\ell_1 \mid H_1)}{P(\ell_1 \mid H_0)} > \frac{P(\ell_1-1 \mid H_1)}{P(\ell_1-1 \mid H_0)} > \cdots > \frac{P(\ell_2 \mid H_1)}{P(\ell_2 \mid H_0)}$, all deriving from $\ell$-LLR strictly increasing property.

## Appendix B
## Proof of Theorem 1

We prove the strictly increasing property of $\ell$-LLR by induction. Let us assume there exists a set of $(t-1)$ sensors with local performances $(P_{D,k}, P_{F,k})$, $k \in \{1,\ldots,t-1\}$. The number of active sensors in this case is denoted $\ell_{t-1} \triangleq \sum_{k=1}^{t-1} x_k$, $\ell_{t-1} \in \mathcal{L}_{t-1}$. We denote the probability of $\ell$ active sensors *out-of-*$(t-1)$, given $H_i$, as $P_{t-1}(\ell \mid H_i)$, $H_i \in \mathcal{H}$ and the corresponding $\ell$-LLR as $\lambda_{t-1}(\ell)$.

*Initialization:* The strictly increasing property of $\ell$-LLR in single sensor case $\lambda_1(\ell_1) > \lambda_1(\ell_1 - 1)$, $\ell_1 \in \mathcal{L}_1\backslash\{0\}$, is straightly verified when $P_{D,1} > P_{F,1}$.

*Induction:* Let us assume that for a specific configuration of $(t-1)$ sensors the $\ell$-LLR $\lambda_{t-1}(\ell_{t-1})$ satisfies the strictly increasing property, that is $\lambda_{t-1}(\ell_{t-1}) > \lambda_{t-1}(\ell_{t-1}-1)$, $\ell_{t-1} \in \mathcal{L}_{t-1}\backslash\{0\}$. If we add the $t$th sensor satisfying $P_{D,t} > P_{F,t}$ and we prove that the new $\ell$-LLR $\lambda_t(\ell_t) > \lambda_t(\ell_t - 1)$, $\ell_t \in \mathcal{L}_t\backslash\{0\}$, i.e., it retains strictly increasing property, then the proof is complete.

To proceed let us first define $a(\ell) \triangleq P_{t-1}(\ell \mid H_1)$, $b(\ell) \triangleq P_{t-1}(\ell \mid H_0)$, $c(\ell) \triangleq P_1(\ell \mid H_1)$ and $d(\ell) \triangleq P_1(\ell \mid H_0)$.

The number of sensors transmitting when the $t$th sensor is added is then given by $\ell_t = \sum_{k=1}^{t} x_k = \ell_{t-1} + x_t$. The pmfs $P_t(\ell_t \mid H_0)$ and $P_t(\ell_t \mid H_1)$ are then given by [18]

$$P_t(\ell_t \mid H_0) = (b * d)(\ell_t) \quad P_t(\ell_t \mid H_1) = (a * c)(\ell_t) \quad (22)$$

The LLR strictly increasing condition is then formulated as follows

$$\exp\left[\lambda_t(\ell_t)\right]=\frac{(a*c)(\ell_t)}{(b*d)(\ell_t)}>\frac{(a*c)(\ell_t-1)}{(b*d)(\ell_t-1)} \\ =\exp\left[\lambda_t(\ell_t-1)\right] \quad (23)$$

By exploiting the support set of $c(\ell)$ and $d(\ell)$ we can rewrite (23) as follows

$$\frac{\sum_{k\in\{0,1\}}c(k)a(\ell_t-k)}{\sum_{k\in\{0,1\}}d(k)b(\ell_t-k)}>\frac{\sum_{k\in\{0,1\}}c(k)a(\ell_t-1-k)}{\sum_{k\in\{0,1\}}d(k)b(\ell_t-1-k)} \quad (24)$$

where obviously $a(t)=b(t)=0$. Exploiting $c(0)+c(1)=(1-P_{D,t})+P_{D,t}=1$, $d(0)+d(1)=(1-P_{F,t})+P_{F,t}=1$, we obtain

$$\frac{[1-c(1)]a(\ell_t)+c(1)a(\ell_t-1)}{[1-d(1)]b(\ell_t)+d(1)b(\ell_t-1)} \\ >\frac{[1-c(1)]a(\ell_t-1)+c(1)a(\ell_t-2)}{[1-d(1)]b(\ell_t-1)+d(1)b(\ell_t-2)} \quad (25)$$

The condition expressed in (25) can be rewritten as:

$$\begin{aligned}&\{[1-c(1)][1-d(1)][a(\ell_t)b(\ell_t-1)-a(\ell_t-1)b(\ell_t)]\}\\&+\{c(1)d(1)[a(\ell_t-1)b(\ell_t-2)-a(\ell_t-2)b(\ell_t-1)]\}\\&+\{c(1)[1-d(1)][a(\ell_t-1)b(\ell_t-1)-a(\ell_t-2)b(\ell_t)]\}\\&-\{[1-c(1)]d(1)[a(\ell_t-1)b(\ell_t-1)\\&-a(\ell_t)b(\ell_t-2)]\}>0 \end{aligned} \quad (26)$$

Since $\frac{a(\ell_t)}{b(\ell_t)}>\frac{a(\ell_t-1)}{b(\ell_t-1)}>\frac{a(\ell_t-2)}{b(\ell_t-2)}$, we have that:

$$[a(\ell_t)b(\ell_t-1)-a(\ell_t-1)b(\ell_t)]>0$$
$$[a(\ell_t-1)b(\ell_t-2)-a(\ell_t-2)b(\ell_t-1)]>0 \quad (27)$$
$$[a(\ell_t)b(\ell_t-2)-a(\ell_t-2)b(\ell_t)]>0 \quad (28)$$

The condition in (26) is satisfied since positivity of the first two terms follows from the inequalities in (27), and the difference of the third and fourth terms in (26) is positive because $c(1)[1-d(1)]>[1-c(1)]d(1)$ (since $P_{D,t}>P_{F,t}$) and exploiting the inequality in (28). This concludes the proof.

## Appendix C
## Proof of Theorem 2

The proof follows in the first part similar steps as in [2]; for this reason we will only sketch it and underline the substantial differences. We use here the *characteristic function* of the vector $\boldsymbol{z}\,|\,H_i$, $i\in\{1,2\}$, denoted as $\Phi_{\boldsymbol{z}\,|\,H_i}(\boldsymbol{t})$, to easily evaluate the limit for $K\to+\infty$. We then use this result, in conjunction with *Levy's Continuity Theorem* [18] to demonstrate the convergence in distribution of large-system $p(\boldsymbol{z}\,|\,H_i)$. Let us now write the characteristic function of $\boldsymbol{z}\,|\,H_0$ as a function of $\boldsymbol{t}=(\boldsymbol{t}_1^t,\boldsymbol{t}_2^t)^t$:

$$\Phi_{\boldsymbol{z}\,|\,H_0}(\boldsymbol{t})=\mathbb{E}_{\boldsymbol{z}\,|\,H_0}\left\{\exp\left(j\boldsymbol{t}_1^t\boldsymbol{z}_1+j\boldsymbol{t}_2^t\boldsymbol{z}_2\right)\right\} \quad (29)$$
$$=\int\cdots\int\exp\left(j\boldsymbol{t}_1^t\boldsymbol{z}_1+j\boldsymbol{t}_2^t\boldsymbol{z}_2\right) \\ \times\sum_{\ell=0}^{K}p(\boldsymbol{z}_1,\boldsymbol{z}_2\,|\,\ell)P(\ell\,|\,H_0)\,d\boldsymbol{z}_1\,d\boldsymbol{z}_2 \quad (30)$$

where $\boldsymbol{z}_1\triangleq\Re\{\boldsymbol{z}\}$ and $\boldsymbol{z}_2\triangleq\Im\{\boldsymbol{z}\}$ (with $\boldsymbol{t}_i$, $i\in\{1,2\}$, representing the index-corresponding dual vectors over Fourier domain). Following analogous steps as in [2], exploiting: *i*) conditional independence assumptions such as $p(\boldsymbol{z}_1,\boldsymbol{z}_2\,|\,\ell)=p(\boldsymbol{z}_1\,|\,\ell)p(\boldsymbol{z}_2\,|\,\ell)$ and $p(\boldsymbol{z}_i\,|\,\ell)=\prod_{s=1}^{N}p(z_{i,s}\,|\,\ell)$, $i\in\{1,2\}$; *ii*) the characteristic function of $x\sim\mathcal{N}(0,\sigma^2)$ is given by $\Phi_x(t)=\exp(-\frac{\sigma^2t^2}{2})$ [18]; we get

$$\Phi_{\boldsymbol{z}\,|\,H_0}(\boldsymbol{t}) \\ =\sum_{\ell=0}^{K}P(\ell\,|\,H_0)\exp\left[-\frac{1}{4}\sum_{s=1}^{N}\left(t_{1,s}^2+t_{2,s}^2\right)\left(\frac{\ell}{NKP_F}+\frac{\sigma_w^2}{\sigma_h^2KP_F}\right)\right] \quad (31)$$

$$=\exp\left[-\frac{1}{4}\sum_{s=1}^{N}\left(t_{1,s}^2+t_{2,s}^2\right)\frac{\sigma_w^2}{\sigma_h^2KP_F}\right] \\ \times\left\{P_F\exp\left[-\frac{1}{4}\sum_{s=1}^{N}\frac{\left(t_{1,s}^2+t_{2,s}^2\right)}{NKP_F}\right]+(1-P_F)\right\}^K \quad (32)$$

where in the last line we exploited $\ell\,|\,H_0\sim\mathcal{B}(K,P_F)$. Also, exploiting similar noteworthy limits as in [2], eventually we have that $\lim_{K\to+\infty}\Phi_{\boldsymbol{z}\,|\,H_0}(\boldsymbol{t})=\exp[-\frac{1}{2}\sum_{s=1}^{N}\frac{(t_{1,s}^2+t_{2,s}^2)}{2N}]$. Applying the Continuity Theorem [18], we obtain $\boldsymbol{z}\,|\,H_0\xrightarrow{d}\mathcal{N}_{\mathbb{C}}(\boldsymbol{0}_N,\frac{1}{N}\boldsymbol{I}_N)$. In a similar way it can be shown that $\boldsymbol{z}\,|\,H_1\xrightarrow{d}\mathcal{N}_{\mathbb{C}}(\boldsymbol{0}_N,\frac{P_D}{P_FN}\boldsymbol{I}_N)$.

The last part consists in proving (14) and (15). The large-system probabilities of false alarm and detection can be expressed in the equivalent form:

$$P_{F_0}^*(\bar{\gamma})=P(\|\boldsymbol{z}\|^2\geq\bar{\gamma}\,|\,H_0)=P\left(\frac{1}{2N}\xi\geq\bar{\gamma}\,|\,H_0\right) \quad (33)$$
$$P_{D_0}^*(\bar{\gamma})=P(\|\boldsymbol{z}\|^2\geq\bar{\gamma}\,|\,H_1)=P\left(\frac{P_D}{2P_FN}\xi\geq\bar{\gamma}\,|\,H_1\right) \quad (34)$$

where $\xi\sim\chi^2_{(2N)}$. The probabilities are then easily calculated evaluating the cumulative distribution function of $\xi$ [18]:

$$P_{F_0}^*(\bar{\gamma})=\int_{2\bar{\gamma}N}^{+\infty}p(\xi)\,d\xi \quad P_{D_0}^*(\bar{\gamma})=\int_{2\bar{\gamma}N\frac{P_F}{P_D}}^{+\infty}p(\xi)\,d\xi \quad (35)$$

which provides the result.

## Acknowledgment

The authors would like to thank Prof. P. K. Willett at University of Connecticut for helpful discussions, as well as the associate editor and the anonymous reviewers for their valuable comments that allowed to improve significantly the correspondence.

## References

[1] C. R. Berger, M. Guerriero, S. Zhou, and P. Willett, "PAC vs. MAC for decentralized detection using noncoherent modulation," *IEEE Trans. Signal Process.*, vol. 57, no. 9, pp. 3562–3575, Sep. 2009.
[2] F. Li, J. S. Evans, and S. Dey, "Decision fusion over noncoherent fading multiaccess channels," *IEEE Trans. Signal Process.*, vol. 59, no. 9, pp. 4367–4380, Sep. 2011.
[3] P. K. Varshney, *Distributed Detection and Data Fusion*, 1st ed. New York: Springer-Verlag, 1996.
[4] B. Chen, L. Tong, and P. K. Varshney, "Channel-aware distributed detection in wireless sensor networks," *IEEE Signal Process. Mag.*, vol. 23, no. 4, pp. 16–26, Jul. 2006.
[5] M. Gastpar, M. Vetterli, and P. Dragotti, "Sensing reality and communicating bits: A dangerous liaison," *IEEE Signal Process. Mag.*, vol. 23, no. 4, pp. 70–83, Jul. 2006.
[6] J. F. Chamberland and V. V. Veeravalli, "Wireless sensors in distributed detection applications," *IEEE Signal Process. Mag.*, vol. 24, no. 3, pp. 16–25, May 2007.
[7] B. Chen, R. Jiang, T. Kasetkasem, and P. K. Varshney, "Channel aware decision fusion in wireless sensor networks," *IEEE Trans. Signal Process.*, vol. 52, no. 12, pp. 3454–3458, Dec. 2004.
[8] R. Niu, B. Chen, and P. K. Varshney, "Fusion of decisions transmitted over Rayleigh fading channels in wireless sensor networks," *IEEE Trans. Signal Process.*, vol. 54, no. 3, pp. 1018–1027, Mar. 2006.

[9] A. Lei and R. Schober, "Coherent Max-Log decision fusion in wireless sensor networks," *IEEE Trans. Commun.*, vol. 58, no. 5, pp. 1327–1332, May 2010.

[10] X. Zhang, H. V. Poor, and M. Chiang, "Optimal power allocation for distributed detection over MIMO channels in wireless sensor networks," *IEEE Trans. Signal Process.*, vol. 56, no. 9, pp. 4124–4140, Sep. 2008.

[11] D. Ciuonzo, G. Romano, and P. Salvo Rossi, "Channel-aware decision fusion in distributed MIMO wireless sensor networks: Decode-and-fuse vs. decode-then-fuse," *IEEE Trans. Wireless Commun.*, vol. 11, no. 8, pp. 2976–2985, Aug. 2012.

[12] M. K. Banavar, A. D. Smith, C. Tepedelenlioglu, and A. Spanias, "Distributed detection over fading MACs with multiple antennas at the fusion center," in *Proc. IEEE ICASSP*, Mar. 2010, pp. 2894–2897.

[13] S. M. Kay, *Fundamentals of Statistical Signal Processing, Volume 2: Detection Theory*. Englewood Cliffs, NJ: Prentice-Hall PTR, Jan. 1998.

[14] L. A. Belfore, "An $O(n(\log_2(n))^2)$ algorithm for computing the reliability of $k$-out-of-$n$: $G$ & $k$-to-$l$-out-of-$n$: $G$ systems," *IEEE Trans. Rel.*, vol. 44, no. 1, pp. 132–136, Mar. 1995.

[15] S. X. Chen and J. S. Liu, "Statistical applications of the Poisson-Binomial and conditional Bernoulli distributions," *Statistica Sinica*, vol. 7, no. 4, 1997.

[16] M. Fernandez and S. Williams, "Closed-form expression for the Poisson-Binomial probability density function," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 46, no. 2, pp. 803–817, Apr. 2010.

[17] H. V. Poor, *An Introduction to Signal Detection and Estimation*. New York: Springer-Verlag, 1994.

[18] A. F. Karr, *Probability*. New York: Springer-Verlag, 1993.